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THE BOOLEAN SPACE OF ORDERINGS OF A FIELD[1]

BY

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ABSTRACT. It has been pointed out by Knebusch, Rosenberg and Ware that the set X of all orderings on a formally real field can be topologized to make a Boolean space (compact, Hausdorff and totally disconnected). They have called the sets of orderings $W(a) = \{< \text{ in } X | a < 0\}$ the Harrison subbasis of X. This subbasis is closed under symmetric difference and complementation. In this paper it is proved that, given any Boolean space X, there exists a formally real field F such that X is homeomorphic to the space of orderings on F. Also, an example is given of a Boolean space and a basis of clopen sets closed under symmetric difference and complementation which cannot be the Harrison subbasis of any formally real field.

1. Introduction and notation. Let F be a formally real field, i.e. one which can be ordered. Harrison (unpublished) and Leicht and Lorenz [12] have shown that the set of orderings $X(F)$ of the field F is in bijective correspondence with the set of minimal prime ideals of the Witt ring $W(F)$, whose elements are equivalence classes of anisotropic quadratic forms over F. Thus $X(F)$ has the topology induced by the Zariski topology of the prime spectrum of $W(F)$. Under this topology, $X(F)$ is a Boolean space (i.e. compact, Hausdorff and totally disconnected). An equivalent way of defining this topology is to take as a subbasis the collection of all subsets of $X(F)$ of the form $W(a) = W_F(a) = \{< \text{ on } F | a < 0\}$, where $a \in \dot{F} = F - \{0\}$ [10]. We shall call this subbasis the Harrison subbasis and denote it by $H(F)$. Note that the complement of $W(a)$ is $W(a)^c = W(-a)$, so these sets are all clopen (both closed and open). All of the above can be generalized from fields to commutative semilocal rings where one considers signatures instead of orderings [8].

The primary goal of this paper is to characterize completely the topological spaces $X(F)$. We have already noted that they must be Boolean spaces. With Theorem 5 we establish the converse: Given any Boolean space one can find a

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formally real field such that the space of orderings of the field is homeomorphic to the given space. The proof involves a technique of constructing fields with given spaces by starting with a given field and building larger fields as composites of infinitely many quadratic extensions. This technique can also be useful for constructing fields with various kinds of specific conditions on the orderings such as our construction of an "order closure" in §4.

In §3 we give an example of a Boolean space together with a basis of clopen sets which cannot be the Harrison subbasis for the space of orderings of any formally real field, and we discuss the implications of the existence of such an example. In §4 we relate the results of §2 to Boolean algebras.

Given two formally real fields $F \subset K$, we obtain a natural mapping $\rho\colon X(K) \longrightarrow X(F)$ which restricts the orderings on K to the subfield F. This canonical restriction mapping is continuous since $\rho^{-1}(W_F(a)) = W_K(a)$ for any $a \in \dot{F}$. In this way we get a functor from the category of formally real fields to the category of Boolean topological spaces.

One other concept which we shall make extensive use of is the *strong approximation property* (*SAP*), originally introduced in [10]. A formally real field F is said to satisfy SAP if $H(F)$ consists of all the clopen subsets of $X(F)$. This is in fact equivalent to the seemingly weaker condition that $H(F)$ be a basis for $X(F)$ [3, Theorem 3.5].

2. **Characterization of the space of orderings.** In this section we shall show that every Boolean space can be realized as the space of orderings of a field. We begin by seeing how we can eliminate orderings of a field by going to extension fields.

LEMMA 1. *If F is an ordered field, $a \in F$, $\sqrt{a} \notin F$ and $a > 0$, then $K = F(\sqrt[2^n]{a}, n = 1, 2, \ldots)$ has a unique ordering which extends the given ordering of F. Since $\sqrt{a} \in K$, a must be positive in all orderings of K. Thus the orderings of K are in bijective correspondence with the orderings of F in which a is positive.*

PROOF. For each $m = 1, 2, 3, \ldots$, let $F_m = F(\sqrt{a}, \sqrt[4]{a}, \ldots, \sqrt[2^m]{a})$. Note that the elements $\sqrt{a}, \sqrt[4]{a}, \ldots, \sqrt[2^{m-1}]{a}$ are all squares in F_m and hence must be positive. Thus by [1, p. 37] and induction on m, we see that each F_m has precisely two orderings extending the given ordering on F: one in which $\sqrt[2^m]{a}$ is positive and one in which it is negative. Since any ordering of $K = \bigcup_{m=1}^{\infty} F_m$ induces an ordering on each F_m, K can have at most one ordering extending the ordering on F, namely the one in which $\sqrt[2^n]{a}$ is positive for all n. But this clearly does define an ordering on K since we have a compatible family of orderings on the fields F_m (take the union of the sets of positive elements in each F_m as the set of positive elements for the ordering of K).

PROPOSITION 2. *Let F be a formally real field, $Y \subset X(F)$ such that Y^c, the complement of Y, is a union of sets in $\mathcal{H}(F)$. Then there exists an algebraic extension K of F such that the canonical map ρ: $X(K) \longrightarrow X(F)$ is a homeomorphism onto Y.*

PROOF. By hypothesis, we may write $Y^c = \bigcup_{a \in A} W(a)$ for some subset $A \subset F$. We may assume no element of A is a square in F since $W(r^2) = \emptyset$ for any r. Let $K = F(\sqrt[2^n]{a},\ a \in A,\ n = 1, 2, 3, \ldots)$. Since each $a \in A$ is a square in K, it must be positive in all orderings of K; i.e. the only orderings of F which extend to K are those in Y, the set of all orderings of F in which all elements of A are positive. Thus $\rho(X(K)) \subset Y$. We need to show that ρ maps $X(K)$ onto Y and that ρ is injective. Since ρ is continuous, [14, Theorem E, p. 131] will then imply ρ is a homeomorphism onto Y.

Let $\mathcal{M}$ be the set of pairs (L, B) where $B \subset A$ and $L = F(\sqrt[2^n]{a},\ a \in B,\ n = 1, 2, \ldots)$ is a subfield of K such that:

(1) ρ_L: $X(L) \longrightarrow X(F)$ is injective;

(2) $Y \subset \rho_L(X(L))$.

Since $(F, \emptyset) \in \mathcal{M}$, the set $\mathcal{M}$ is nonempty. The set $\mathcal{M}$ is partially ordered by inclusion on the subsets of A. Note that if (L_1, B_1) and (L_2, B_2) are in $\mathcal{M}$ with $B_1 \subset B_2$, then the following diagram commutes

$$\begin{array}{ccc} X(L_2) & \longrightarrow & X(L_1) \\ \downarrow & & \downarrow \\ X(F) & = & X(F) \end{array}$$

Let $\{(L_\alpha, B_\alpha)\}$ be a simply ordered subset of $\mathcal{M}$ and set $L = \bigcup L_\alpha$, $B = \bigcup B_\alpha$. Then $L = F(\sqrt[2^n]{a},\ a \in B,\ n = 1, 2, \ldots)$ and (1) and (2) hold: ρ_L is injective since each ordering of F which extends to L extends uniquely to each L_α, hence extends uniquely to $L = \bigcup L_\alpha$. We have $Y \subset \rho_L(X(L))$ since each ordering in Y extends to each L_α and hence extends to $L = \bigcup L_\alpha$. Therefore $(L, B) \in \mathcal{M}$. By Zorn's lemma $\mathcal{M}$ has a maximal element (L_0, B_0). Assume $L_0 \neq K$. Then there exists an element $a_0 \in A$, $a_0 \notin B_0$. By Lemma 1, the field $L_0(\sqrt[2^n]{a_0},\ n = 1, 2, \ldots)$ satisfies conditions (1) and (2), so the pair $(L_0(\sqrt[2^n]{a_0},\ n = 1, 2, \ldots),\ B_0 \cup \{a_0\})$ is in $\mathcal{M}$, a contradiction of the maximality of (L_0, B_0). Therefore $L_0 = K$, so $Y \subset \rho(X(K))$. As was pointed out above, $\rho(X(K)) \subset Y$, hence $Y = \rho(X(K))$. By (1), ρ is injective, so we are done.

COROLLARY 3. *Let F be a field satisfying SAP and let Y be any closed subspace of $X(F)$. Then there exists an algebraic extension K of F such that the* *canonical map ρ: $X(K) \longrightarrow X(F)$ is a homeomorphism onto Y. In this case K also satisfies SAP.*

PROOF. Since Y is closed, Y^c is open. Since F satisfies SAP, the family of sets $H(F)$ is a basis and Y^c is a union of sets in $H(F)$. Hence Proposition 2 applies. K satisfies SAP since F does and ρ is injective [2, Theorem 2].

A topological space is said to be *second countable* if it has a basis consisting of a countable number of sets. We can now prove a special case of our main theorem.

COROLLARY 4. *Any second countable Boolean space is homeomorphic to the space of orderings of an algebraic extension of* **Q**, *the field of rational numbers.*

PROOF. Let $F = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots)$, the rational numbers extended by the square roots of all the rational primes. The field F satisfies SAP by [8, Remark 3.22] or [13, Example 2.10, Chapter 3]. We shall show that $X(F)$ is homeomorphic to the Cantor set C. Once this is demonstrated, we will be done by Corollary 3 since any second countable Boolean space can be embedded as a closed subset of the Cantor set [7, p. 100].

Every ordering of F is induced by an embedding of F into **R** [13, Example 2.4, Chapter 3]. The embeddings of F into **R** are determined by specifying whether each $\sqrt{p}$ goes to $+\sqrt{p}$ or $-\sqrt{p}$ in **R**. Since C is a countable product of discrete two point spaces with the product topology, we can represent the elements of C by infinite sequences $(c_1, c_2, c_3, \ldots)$, $c_i \in \{0, 1\}$. We thus have a set theoretical bijection f: $X(F) \longrightarrow C$ defined by $f(<) = (c_1, c_2, c_3, \ldots)$ where $c_i = 0$ if $\sqrt{p_i}$ is positive in the ordering $<$, and $c_i = 1$ if $\sqrt{p_i}$ is negative in the ordering $<$, where p_i denotes the ith prime number. Since $X(F)$ and C are compact and Hausdorff, we need only show that f is continuous [14, Theorem E, p. 131]. To see that f is continuous, let U be any element of the standard basis for the product topology on C; i.e.

$$U = \{(c_1, c_2, \ldots) \in C \mid c_{i_1} = c_{i_2} = \ldots = c_{i_n} = 0, \quad c_{j_1} = c_{j_2} = \ldots = c_{j_m} = 1\}.$$

Then $f^{-1}(U)$ equals the set of orderings on F in which $\sqrt{p_{i_1}}, \sqrt{p_{i_2}}, \ldots, \sqrt{p_{i_n}}$ are positive and $\sqrt{p_{j_1}}, \sqrt{p_{j_2}}, \ldots, \sqrt{p_{j_m}}$ are negative; i.e.

$$f^{-1}(U) = W(-\sqrt{p_{i_1}}) \cap \ldots \cap W(-\sqrt{p_{i_n}}) \cap W(\sqrt{p_{j_1}}) \cap \ldots \cap W(\sqrt{p_{j_m}}).$$

Thus $f^{-1}(U)$ is open in $X(F)$ and f is continuous.

REMARK. The converse of Corollary 4 is also true. If F is a formally real algebraic extension of **Q**, then F is a countable field, hence $H(F)$ is countable. Thus $X(F)$ is a second countable Boolean space.

We shall now prove the main theorem of this section.

THEOREM 5. *Every Boolean space X is homeomorphic to a space of orderings $X(F)$ for some formally real field F.*

PROOF. For each infinite cardinal number $\mathfrak{m}$, let $D_{\mathfrak{m}}$ denote the Cantor cube of weight $\mathfrak{m}$; i.e. the product of $\mathfrak{m}$ copies of the discrete two point space $\{0, 1\}$. The Boolean space X can be embedded as a closed subset of $D_{\mathfrak{m}}$ for a sufficiently large cardinal number $\mathfrak{m}$ [5, Theorem 11, p. 252]. By Corollary 3 we will be done if we prove that for each $\mathfrak{m}$ there exists a field F satisfying SAP such that $X(F)$ is homeomorphic to $D_{\mathfrak{m}}$.

In [6] fields F, F_0 and F_1 are constructed for each cardinal number $\mathfrak{m}$ with the following properties:

(a) F_0 has a unique ordering.

(b) F_1 is ordered.

(c) $F_0 \subset F_1$ and is dense in F_1.

(d) F_1 contains sets of elements $\{\eta_\alpha\}_{\alpha<\mathfrak{m}}$ and $\{\xi_\alpha\}_{\alpha<\mathfrak{m}}$ (where α ranges over the ordinal numbers and $\mathfrak{m}$ is associated with the smallest ordinal in its class) such that $\alpha < \beta < \mathfrak{m}$ implies $\eta_\alpha < \eta_\beta$, $\xi_\alpha < \xi_\beta$; also $\eta_\alpha < 0$, $\xi_\alpha > 0$ for all $\alpha < \mathfrak{m}$; and the set $\{\eta_\alpha\}_{\alpha<\mathfrak{m}} \cup \{\xi_\alpha\}_{\alpha<\mathfrak{m}}$ is algebraically independent over F_0.

(e) F is an algebraic extension of $F_0(\{x_\alpha\}_{\alpha<\mathfrak{m}})$, the x_α being indeterminates over F_0.

(f) Each embedding φ of $F_0(\{x_\alpha\})$ into F_1 given by sending each x_α to η_α or ξ_α in F_1 induces an ordering on $F_0(\{x_\alpha\})$ which extends uniquely to F, and all orderings of F arise in this way.

If $\mathfrak{m}$ is the first infinite cardinal, then F_0 and F_1 can be taken to be **Q** and **R** respectively. In the general case model theory is used to construct F_0 and F_1.

Define $h\colon X(F) \longrightarrow D_{\mathfrak{m}}$ as follows: Let $\varphi_<\colon F_0(\{x_\alpha\}) \longrightarrow F_1$ be the embedding corresponding to an ordering $<$ of F; define

$$h(<)_\alpha = \begin{cases} 0 & \text{if } \varphi_<(x_\alpha) = \eta_\alpha < 0, \\ 1 & \text{if } \varphi_<(x_\alpha) = \xi_\alpha > 0. \end{cases}$$

Condition (f) implies h is bijective. To see that h is continuous, let $U \subset D_{\mathfrak{m}}$ be any basic open set in the product topology, say $U = \Pi_{\alpha<\mathfrak{m}} U_\alpha$ where $U_{\alpha_i} = \{0\}$, $i = 1, \ldots, r$; $U_{\beta_i} = \{1\}$, $i = 1, \ldots, s$; and $U_\alpha = \{0, 1\}$ for $\alpha \neq \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$. Then

$$h^{-1}(U) = W(x_{\alpha_1}) \cap \ldots \cap W(x_{\alpha_r}) \cap W(-x_{\beta_1}) \cap \ldots \cap W(-x_{\beta_s}),$$

an open set in $X(F)$. By [14, Theorem E, p. 131], the map h is a homeomorphism.

Finally, we must show that F satisfies SAP. In view of the homeomorphism above, sets of the form $h^{-1}(U)$ form a basis of $X(F)$. We shall show these sets all lie in $H(F)$, thus making $H(F)$ a basis. Let $V = h^{-1}(U)$ as above. Since $\xi_\alpha > 0$ and $\eta_\alpha < 0$ for all α, property (c) above implies that there exists $\gamma \in F_0$ such that

$$-\sum_{i=1}^{r} \eta_{\alpha_i} + \sum_{i=1}^{s} \xi_{\beta_i} > \gamma$$

$$> \max_{j,k}\left(-\xi_{\alpha_j} - \sum_{i=1;i\neq j}^{r} \eta_{\alpha_i} + \sum_{i=1}^{s} \xi_{\beta_i}, \eta_{\beta_k} - \sum_{i=1}^{r} \eta_{\alpha_i} + \sum_{i=1;i\neq k}^{s} \xi_{\beta_i}\right).$$

Set $a = \Sigma_{i=1}^{r} x_{\alpha_i} - \Sigma_{i=1}^{s} x_{\beta_i} + \gamma$, an element of $F_0(\{x_\alpha\})$. It is then a straightforward computation to check that $\varphi_<(a)$ is negative in F_1 if and only if $< \in V$; i.e. $V = W_F(a)$, $a \in \dot{F}$, so $V \in H(F)$, and the proof is complete.

Remark. The following is another way to construct fields whose spaces of orderings are the spaces $D_{\mathfrak{m}}$. This method is based on the ideas used in Corollary 4 and a result in [4]. Let K be a real closed field of cardinality $\mathfrak{m}$; e.g. K could be the real closure with respect to any ordering of $\mathbf{Q}(\{x_\alpha\}_{\alpha<\mathfrak{m}})$, the x_α being algebraically independent over $\mathbf{Q}$. Fix any ordering of $K(x)$, the rational function field in one variable over K. (For further information on orderings of rational function fields see [2].) Consider the field $F = K(x)(\sqrt{p_0}, \sqrt{p_1}, \ldots)$ where $\{p_\alpha | \alpha < \mathfrak{m}\}$ consists of all linear polynomials over K with leading coefficient ± 1 such that $p_\alpha > 0$ in the fixed ordering of $K(x)$. Then each ordering of F restricts to the fixed ordering of $K(x)$ [2, Lemma 7], so a proof similar to that of Corollary 4 will show that $X(F)$ is homeomorphic to $D_{\mathfrak{m}}$. The field F satisfies SAP by [4, §5]. We wish to thank A. Prestel for pointing out the above construction.

3. **The Harrison subbasis.** The Harrison subbasis $H(F)$ is considerably more important than we have yet seen. We have already noted that $H(F)$ is closed under taking complements. It is also closed under the operation of symmetric difference which we shall denote by +. In fact, $W(a) + W(b) = W(ab)$ since ab is negative if and only if a or b is negative but not both. Thus $H(F)$ is a group with identity $W(1) = \varnothing$. Furthermore, $H(F)$ is isomorphic to the multiplicative group of nonzero elements of F modulo sums of squares [2, Theorem 5].

It is quite possible for two fields F and K to have homeomorphic spaces of orderings but considerably different Harrison subbases. For example,

$F = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots)$ and $K = \mathbf{Q}(x)$ both have spaces homeomorphic to the Cantor set, but F satisfies SAP while K does not [2, Theorem 15].

The example of this section shows that given a Boolean space X and a subbasis $\mathcal{S}$ closed under complementation and symmetric difference, a field F may not exist with a homeomorphism of $X(F)$ to X which carries $\mathcal{H}(F)$ to $\mathcal{S}$. We shall in fact construct a basis $\mathcal{S}$ for the Cantor set $\mathcal{C}$ which consists of clopen sets, is closed under complementation and symmetric difference, but does not contain all of the clopen subsets of $\mathcal{C}$. As was pointed out in §1, this implies $\mathcal{S}$ can never be a Harrison subbasis since for $\mathcal{H}(F)$ to be a basis implies F satisfies SAP which implies $\mathcal{H}(F)$ contains all the clopen subsets of $X(F)$ [3, Theorem 3.5]. It is not difficult to show that if the Boolean space X is countable, any basis of clopen sets for X which is an additive group and contains X must contain all the clopen subsets of X. The following example shows that the result in [3] requires more than just the topology of the situation in the general case.

A further conclusion implied by the existence of such an example is that there exist abstract Witt rings [9] which cannot be realized as Witt rings of fields; this follows from the bijective correspondence between abstract Witt rings and subbases of clopen sets which are closed under symmetric difference and complementation [8, Proposition 3.8].

Example. There exists a basis $\mathcal{S}$ of clopen sets for $\mathcal{C}$ which is closed under complementation and is an additive group under symmetric difference but does not contain all the clopen subsets of $\mathcal{C}$.

We shall think of $\mathcal{C}$ as the set of all infinite sequences of elements from $\{0, 1\}$, a countable product of discrete two point spaces with the product topology. Let $\mathcal{E}$ be the collection of all clopen subsets of $\mathcal{C}$.

We shall use the following notation. Let $v = (a_1, \ldots, a_n)$, $a_i \in \{0, 1\}$, n any positive integer. We denote $\{(x_1, x_2, \ldots) \in \mathcal{C} | x_i = a_i, i = 1, \ldots, n\}$ by X_v. We call n the *length* of X_v. Since X_v is an element of the standard basis for the product topology, it is a clopen set. Furthermore, it is not hard to see that the collection $\mathcal{B}$ of all X_v's is a basis for $\mathcal{C}$. For each v, let $T_v = X_{(v,0)} \cup X_{(v,1,1)}$, which equals $X_{(v,0)} + X_{(v,1,1)}$ since the sets $X_{(v,0)}$ and $X_{(v,1,1)}$ are disjoint; then $T_v \in \mathcal{E}$. We define $\mathcal{S}$ to be the additive subgroup of $\mathcal{E}$ generated by $\{\mathcal{C}; T_v \text{ for all } v\}$. Then $\mathcal{S}$ is a collection of clopen sets closed under complementation as well as symmetric difference since $U^c = U + \mathcal{C} \in \mathcal{S}$ for any $U \in \mathcal{S}$. An examination of these definitions shows that for any v we have $X_v = T_v \cup T_{(v,1)}$. Since $\mathcal{B}$ is a basis, this implies that $\mathcal{S}$ is also a basis. All that remains is to show that $\mathcal{S} \neq \mathcal{E}$; we shall show that the clopen set $X_{(0)} \notin \mathcal{S}$.

Suppose $X_{(0)} \in \mathcal{S}$. Then $X_{(0)} = \Sigma T_{v_i}$ or $X_{(0)} = \mathcal{C} + \Sigma T_{v_i}$ by definition of S. We may assume all v_i are distinct since $T_v + T_v = \varnothing$. Replacing each T_{v_i} by its corresponding sum $X_{(v_i,0)} + X_{(v_i,1,1)}$, we obtain

$$(*) \qquad X_{(0)} = \sum X_{u_i},$$

where all u_i are distinct since the definition of T_v implies no X_u can occur in two different T_{v_i}. Choose j so that X_{u_j} has maximum length among the X_{u_i}. Since each T_v has the form $X_{(v,0)} + X_{(v,1,1)}$, the set X_{u_j} must have the form $X_{(v,1,1)}$; say $X_{u_j} = X_{(t,1,1)}$ where $t = (a_1, \ldots, a_n)$, so that X_{u_j} has length $n + 2$. Note that $X_{(t,1,0)}$ is not in the sum (*) since, if it were, it would be part of a T_v where the other part would be $X_{(t,1,1,1)}$ of length $n + 3$, a contradiction of our choice of j. For each i, we can write $X_{u_i} = \Sigma X^{(i)}_{w_k}$ where the $X^{(i)}_{w_k}$ are pairwise disjoint and of length $n + 2$. We do this by repeatedly applying the relation $X_u = X_{(u,0)} + X_{(u,1)}$ to increase lengths until all the summands reach the maximum length, $n + 2$. The right-hand side of (*) thus becomes a sum of X_v's, all of length $n + 2$. The elements $X_{(t,1,1)}$ and $X_{(t,1,0)}$ may each occur in this new sum several times. The process of increasing lengths will add both of them for each X_{u_i}, $i \neq j$, such that the m_i-tuple u_i consists of the first m_i elements in the $(n + 2)$-tuple u_j. In particular $X_{(t,1,1)}$ and $X_{(t,1,0)}$ will be added to the new sum the same number of times. Since $X_{(t,1,1)}$ occurs in (*) and $X_{(t,1,0)}$ does not, the element $X_{(t,1,1)}$ will occur one more time than $X_{(t,1,0)}$ in the new sum where all summands have the same length. When we cancel X_v's in pairs by the symmetric difference operation, we obtain a disjoint sum of X_v's all of length $n + 2$ which includes either $X_{(t,1,1)}$ or $X_{(t,1,0)}$ but not both. But this clearly cannot equal $X_{(0)}$. Hence $X_{(0)} \notin S$.

4. **Boolean algebras.** It is a well-known fact that the category of Boolean algebras and the category of Boolean spaces are equivalent under the correspondence which takes a Boolean space to the Boolean algebra of clopen subsets of the space and takes a Boolean algebra to its Stone space of maximal ideals [15]. In this section we shall present the results of §2 in terms of Boolean algebras and the corresponding Boolean rings determined by the Boolean algebras.

We remarked in §3 that the group $H(F)$ of subbasic open sets under the operation of symmetric difference is isomorphic to the multiplicative group $\dot{F}/\sigma(F)$ where $\sigma(F)$ denotes the subgroup of nonzero sums of squares in F [2, Theorem 5]. In general, the group $\dot{F}/\sigma(F)$ is less interesting than $\dot{F}/\dot{F}^2$, the group of nonzero square classes of F. A field is said to be *pythagorean* if $\sigma(F) = \dot{F}^2$. In order to state our results in terms of $\dot{F}/\dot{F}^2$, we must extend them a little further, beginning with the following definition.

DEFINITION. Let F be any formally real field. We call an algebraic extension K of F an *order closure* of F if the restriction mapping $\rho: X(K) \longrightarrow X(F)$ is a homeomorphism and K is maximal with respect to this property.

Thus if F has a unique ordering, an order closure is a real closure of F. To prove the existence of order closures in general, we shall use

LEMMA 6. *Assume F is a formally real field and $F = \varinjlim F_\alpha$ where $\{F_\alpha\}$ is a collection of subfields of F. Then the canonical maps $\rho_\alpha: X(F) \longrightarrow X(F_\alpha)$ induce a homeomorphism $\varphi: X(F) \longrightarrow \varprojlim X(F_\alpha)$.*

PROOF. The maps ρ_α induce a continuous function $\varphi: X(F) \longrightarrow \varprojlim X(F_\alpha)$ by definition of inverse limit. We need only show that φ is bijective since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism [14, Theorem E, p. 131]. But φ is surjective since any compatible family of orderings on the fields F_α induces an ordering on F (for the set of positive elements of F take the union of the sets of positive elements in each F_α). And φ is injective since given any two distinct orderings on F, there exists an element $a \in F$ which is positive in one ordering and negative in the other; the element a is contained in some cofinal collection of F_α's, hence the two orderings have distinct images in $\varprojlim X(F_\alpha)$.

THEOREM 7. *Let F be any formally real field. Then F has an order closure and all order closures of F are pythagorean.*

PROOF. Existence of order closures follows from a simple application of Zorn's lemma since Lemma 6 implies that given any simply ordered chain of fields K_i containing F with $\rho_i: X(K_i) \longrightarrow X(F)$ a homeomorphism for each i, the union $K = \bigcup K_i$ again has $\rho: X(K) \longrightarrow X(F)$ a homeomorphism.

Let K be an order closure of F and let $y \in K$. To show that K is a pythagorean field, it is sufficient to show that $\sqrt{1+y^2}$ lies in K [11, Chapter 8, §3]. Since $1 + y^2$ is always positive, Lemma 1 implies that the restriction mapping

$$X(K(\sqrt[2^n]{1+y^2}, n = 1, 2, \ldots)) \longrightarrow X(K) \longrightarrow X(F)$$

is a homeomorphism. Thus the maximality of K implies that $\sqrt{1+y^2} \in K$.

REMARK. (a) The reason we consider an order closure rather than pythagorean closure is that in general the pythagorean closure of a field F has many orderings extending each ordering of F. For example, the pythagorean closure of **Q** has an uncountable number of orderings.

(b) For our purposes below we do not actually need an order closure, but only a pythagorean algebraic extension to which every ordering extends uniquely.

Such a field can be constructed by emulating the usual construction of the pythagorean closure [11, Chapter 8, §3] but adjoining all 2^nth roots of $1 + y^2$ at each stage rather than just the square root. In general this procedure will give a smaller field than an order closure.

THEOREM 8. *Given any Boolean algebra* $\mathcal{A}$, *there exists a formally real field* K *such that* $X(K)$ *is homeomorphic to the Stone space* X *of* $\mathcal{A}$; *furthermore,* K *can be chosen so that*

(a) $\mathcal{H}(K)$ *is closed under the Boolean algebra operations of intersection and union, in which case* $\mathcal{H}(K)$ *and* $\mathcal{A}$ *are isomorphic as Boolean algebras*; *and*

(b) *the additive group structure of the Boolean ring determined by* $\mathcal{A}$ *is isomorphic to the multiplicative group* $\dot{K}/\dot{K}^2$ *of nonzero square classes of* K.

PROOF. Choose F as in Theorem 5. Then $X(F)$ is homeomorphic to X. By the proof of that theorem and Corollary 3, the field F can be chosen to satisfy SAP; i.e. $\mathcal{H}(F)$ equals the Boolean algebra of all clopen subsets of $X(F)$, hence is isomorphic to $\mathcal{A}$ by the Stone representation theorem [15]. Now let K be an order closure of F. Then $X(K) \cong X(F) \cong X$. Also K satisfies SAP [2, Theorem 2], so $\mathcal{H}(K)$ is isomorphic to $\mathcal{A}$. Since K is pythagorean, the group $\dot{K}/\dot{K}^2 = \dot{K}/\sigma(K)$ is isomorphic to $\mathcal{H}(K)$ under the operation of symmetric difference, which is isomorphic to the additive group structure of the Boolean ring determined by $\mathcal{A}$.

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